

## The Principle of Mathematical Induction

Let  $a$  be the first value of  $n$  to be considered and let  $P(n)$  be a predicate with  $n$  as the predicate variable.

IF

(1)  $P(a)$  is true (i.e,  $P(n)$  when  $n$  is set to its first value,  $a$ ) and

(2) For every integer  $k \geq a$ ,

If  $P(k)$  is true, Then  $P(k+1)$  is true,

THEN

$P(n)$  is true for every integer  $n \geq a$ .

In a proof by Mathematical Induction, the whole proof involves proving that the "IF" conditions of the Principle of Mathematical Induction shown above are true.

The BASIS STEP is the part of the proof which proves condition (1).

The INDUCTIVE STEP is the part of the proof which proves condition (2).

After (1) and (2) have been proved, we can invoke the Principle of Mathematical Induction to conclude: " $\therefore P(n)$  for every integer  $n \geq a$  by the Principle of Mathematical Induction."

Mathematical Induction (in Diagram Form with  $a = 1$ ):

Given an infinite list of statements:

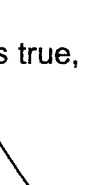
$P_1, P_2, P_3, P_4, \dots$

IF we prove :

that this is true, and that, for all integers  $k \geq 1$ ,



If this is true,



then this is true,



$P_1, P_2, P_3, P_4, \dots, P_k, P_{k+1}, \dots$

THEN ...

If we prove: (1)  $P_1$ , and

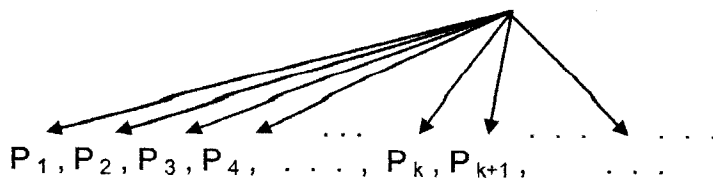
If we prove: (2) For all integers  $k \geq 1$ ,  
if  $P_k$ , then  $P_{k+1}$ ,

Then, we can conclude:

For all integers  $n \geq 1$ ,  $P_n$ ,

by the Principle of Mathematical Induction.

We can logically conclude that ALL of these statements are true.



The Design for Proofs Using Mathematical Induction (1<sup>st</sup> Principle)  
 (  $a$  represents a particular integer.)

To Prove: For every integer  $n$  such that  $n \geq a$ , predicate  $P(n)$ .

Proof: (by Mathematical Induction)

[ The Basis Step shows that  $P(n)$  is true when  $n$  is replaced by  $a$ , the first value of  $n$ .  
 Often,  $a = 1$  or  $a = 0$ . ]

Let  $n = a$  .  
 . . . (calculations with  $n = a$ ) . . . ;  
 $\therefore$  For  $n = a$ ,  $P(n)$  . [ End of Basis Step ]

[The Inductive Step proves that, for every integer  $k$  such that  $k \geq a$ , **if**  $P(k)$ , **then**  $P(k+1)$  . ]

Let  $k$  be any integer such that  $k \geq a$  .  
 Suppose  $P(k)$ . [ Inductive Hypothesis ]  
 [ N.T.S.:  $P(k+1)$  . ] [ That is, we N.T.S. that  $P(n)$  is true when  $n$  is replaced by  $k+1$ . ]  
 . . .  
 . . . (proof statements, one of which is justified by the Inductive Hypothesis supposition)  
 . . .  
 $\therefore P(k+1)$  .  
 $\therefore$  For every integer  $k$  such that  $k \geq a$ , **if**  $P(k)$ , **then**  $P(k+1)$  by Direct Proof.  
 [ End of Inductive Step ]

$\therefore$  For every integer  $n$  such that  $n \geq a$ ,  $P(n)$ , by the Principle of Mathematical Induction.  
 QED

The last statement shown here in the Inductive Step,

"For every integer  $k$  such that  $k \geq a$ , **if**  $P(k)$ , **then**  $P(k+1)$ , by Direct Proof, "

is **required** by Dr. Shirley to be written as the last statement of any Inductive Step (because it says that that the Inductive Step has validly served its purpose and has done so using the method of Direct Proof). Dr. Shirley refers to it as "**The Required Last Statement of the Inductive Step**" when he is grading these proofs.

Also, the statement "**Suppose  $P(k)$** " in the Inductive step is called *the Inductive Hypothesis*. The one and only required comment in this class is one which identifies this statement as being the Inductive Hypothesis of the proof: "**Suppose  $P(k)$ . [ Inductive Hypothesis ]**"

Whenever a conclusion is justified by this supposition, this is indicated by saying:

"  $\therefore$  Such-and-such, **by the Inductive Hypothesis**. "

Note: The first statement of the Inductive Step, shown above, has the wording

"Let  $k$  be any integer such that  $k \geq a$ ,"

Using **this** wording of the sentence to begin the Inductive Step is also **required**.

The author of the textbook uses a different wording of the sentence to begin the Inductive Step. The wording that the author uses most often to begin an Inductive Step is:

"Suppose that  $P(k)$  is true for some integer  $k \geq a$ ." } **NOT ALLOWED**  
 [ We must show that  $P(k+1)$  is true.] "

You may not use this wording to begin the Inductive Step. You must use the required wording "**Let  $k$  be any integer such that  $k \geq a$** " to begin the Inductive Step.

### A First Example of a Proof using the method of Mathematical Induction

Theorem 5.2.2: For every integer  $n \geq 1$ ,  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ .

Proof: (by Mathematical Induction)

[ Basis Step ] Let  $n = 1$ .  $\therefore 1 + 2 + 3 + \dots + n = 1$ , and

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1.$$

$\therefore$  For  $n = 1$ ,  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  by substitution. [End of the Basis Step]

[ The Inductive Step here proves the statement "For every integer  $k \geq 1$ ,  
 if  $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$ ,  
 then  $1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$ . ] [ <-- Big Comment ]

[ Inductive Step ]

Let  $k$  be any integer such that  $k \geq 1$ .

Suppose that  $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$ . [ Inductive Hypothesis ]

[ We need to show that  $1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$ .  
 By the Inductive Hypothesis, we can substitute  $\frac{k(k+1)}{2}$  for  $1 + 2 + 3 + \dots + k$ . ] [ <--Big Comment ]

$$\text{Since } 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2},$$

$$\begin{aligned} \therefore (1 + 2 + 3 + \dots + k) + (k+1) &= \frac{k(k+1)}{2} + (k+1) \text{ by the Inductive Hypothesis and subst.,} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2}, \text{ by rules of Algebra.} \end{aligned}$$

$$\therefore 1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)((k+1) + 1)}{2}, \text{ by transitivity.}$$

$$\begin{aligned} \therefore \text{ For all integers } k \geq 1, \text{ if } 1 + 2 + 3 + \dots + k &= \frac{k(k+1)}{2}, \\ \text{ then } 1 + 2 + 3 + \dots + k + (k+1) &= \frac{(k+1)((k+1) + 1)}{2}, \text{ by Direct Proof.} \\ & \text{ [ End of Inductive Step ]} \end{aligned}$$

Therefore, for every integer  $n \geq 1$ ,  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$   
by the Principle of Mathematical Induction. Q E D

Notes:

A) In a comment, you must identify the statement which supposes that  $P(k)$  is true as being the Inductive Hypothesis.

The step at which this Inductive Hypothesis supposition is applied must have, as part of its justification, the phrase **by the Inductive Hypothesis**.

B) In verifying that  $1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)((k+1) + 1)}{2}$

the proof correctly uses a string of equalities which

1) **begins with the expression on the left** (namely, " $1 + 2 + 3 + \dots + k + (k+1)$ "),

2) presents a number of intermediate expressions, each one of which is clearly equivalent to the previous expression, and

3) **ends with the expression on the right** (namely,  $\frac{(k+1)((k+1) + 1)}{2}$ ).

**Use that method for verifying equations!**

Do not use the unacceptable technique that Dr. Shirley calls "Equation Balancing", in which the first statement is stating the equation that is to be proved, followed a string of equivalent simpler equations, leading to an equation with the same expression on both the left and the right sides.

**This "Equation Balancing" technique for proving an equation is NOT ALLOWED.**

An example of this unacceptable technique proceeds as follows:

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)((k+1) + 1)}{2}$$

$$(1 + 2 + 3 + \dots + k) + (k+1) = \frac{(k+1)((k+1) + 1)}{2}$$

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+2)(k+1)}{2}$$

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2}$$

$$\frac{(k+2)(k+1)}{2} = \frac{(k+2)(k+1)}{2} \quad \text{"Check!"}$$

The reason that this is incorrect and unacceptable is that every statement in a proof (which is not a definition or a supposition) must be a true conclusion, deduced from statements made previously in the proof and from the results of theorems proved earlier.

In the list of statements in the "Equation Balancing" method, the first statement cannot be deduced as being true from the previous statements.

Because the method of "Equation Balancing" breaks the "Ground Rules of Proof Writing" in this way, "Equation Balancing" is not an acceptable method for proving the truth of an equation.

**DON'T USE IT!**

# A SECOND EXAMPLE OF A PROOF USING THE METHOD OF MATH'LE ENO'N

## A SIMPLE INDUCTION PROOF ABOUT A SEQUENCE

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The sequence  $(d_n)$  is defined as follows:

$$d_1 = 3 \text{ and, for every integer } t \geq 2, d_t = 2d_{t-1} + 1.$$

To Prove: For every integer  $n \geq 1$ ,  $d_n = 2^{(n+1)} - 1$ .

Proof:

[Basis Step] Let  $n = 1$ .  $d_n = d_1 = 3$ , by the sequence definition.

$$2^{(n+1)} - 1 = 2^{(1+1)} - 1 = 2^2 - 1 = 4 - 1 = 3$$

$\therefore$  For  $n = 1$ ,  $d_n = 2^{(n+1)} - 1$ , by substitution.

[INDUCTIVE STEP]

Let  $k$  be any integer such that  $k \geq 1$ .

Suppose  $d_k = 2^{(k+1)} - 1$ . [Inductive Hypothesis]

[WTS:  $d_{k+1} = 2^{(k+1)+1} - 1$ .]

SINCE  $k \geq 1$ ,  $k+1 \geq 2$ .

[We need to verify the formula  $d_t = 2d_{t-1} + 1$  applies to  $d_{k+1}$ .]

SINCE  $k+1 \geq 2$ ,  $d_{k+1} = 2d_k + 1$ , by the sequence def'n.

Recall that  $d_k = 2^{(k+1)} - 1$ , by the Inductive Hypothesis.

$\therefore d_{k+1} = 2[2^{(k+1)} - 1] + 1$ , by substitution,

$$= 2 \cdot 2^{(k+1)} - 2 + 1 = 2^{(k+2)} - 1.$$

$\therefore d_{k+1} = 2^{(k+1)+1} - 1$ . [Which is what we needed to show.]

$\therefore$  FOR ALL INTEGERS  $k \geq 1$ , if  $d_k = 2^{(k+1)} - 1$ , then  $d_{k+1} = 2^{(k+1)+1} - 1$ , by Direct Proof.

[END OF INDUCTIVE STEP]

$\therefore$  By Mathematical Induction,  $d_n = 2^{(n+1)} - 1$ , for all integers  $n \geq 1$ . QED

### A Third Example of a Proof using the method of Mathematical Induction

To Prove: For every integer  $n$  such that  $n \geq 1$ ,  $(n^3 - n)$  is divisible by 3.

Proof: (by Mathematical Induction)

[Basis Step]

Let  $n = 1$ . Thus,  $(n^3 - n) = (1^3 - 1) = 0$  and 0 is divisible by 3, because  $0 = (3)(0)$ .

$\therefore$  For  $n = 1$ ,  $(n^3 - n)$  is divisible by 3. [ End of Basis Step ]

[ The Inductive Step proves "For all integers  $k \geq 1$ ,  
if  $(k^3 - k)$  is divisible by 3, then  $((k+1)^3 - (k+1))$  is divisible by 3." ] [ <-- Big Comment ]

[ Inductive Step ]

Let  $k$  be any integer such that  $k \geq 1$ .

Suppose that  $(k^3 - k)$  is divisible by 3. [ The Inductive Hypothesis ]

[ We need to show that  $((k+1)^3 - (k+1))$  is divisible by 3. ]

By the Inductive Hypothesis,  $3 \mid (k^3 - k)$ .

$\therefore$  By definition of "divisibility",  $(k^3 - k) = 3t$  for some integer  $t$ .

$\therefore k^3 = k + 3t$ . { *Note: When the predicate is one which asserts divisibility, solving for the highest power of  $k$  (or fastest-growing expression) in terms of  $t$  and the other terms (here, the lower powers of  $k$ ) is usually the next step after applying the Inductive Hypothesis.* } [ <-- Big Comment ]

$$\begin{aligned} \therefore (k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - k - 1 \\ &= k^3 + 3k^2 + 2k \quad \text{[ Next, substituting } (k + 3t) \text{ for } k^3 \text{]} \\ &= (k + 3t) + 3k^2 + 2k, \text{ by substitution,} \\ &= 3t + 3k^2 + 3k. \end{aligned}$$

$\therefore (k+1)^3 - (k+1) = 3(t + k^2 + k)$ , by Rules of Algebra and transitivity.

$\therefore (k+1)^3 - (k+1) = 3m$ , where  $m = (t + k^2 + k)$ , which is an integer.

$\therefore 3 \mid ((k+1)^3 - (k+1))$ , that is,  $((k+1)^3 - (k+1))$  is divisible by 3, by definition of "divisible".

$\therefore$  For all integers  $k$  such that  $k \geq 1$ ,

if  $(k^3 - k)$  is divisible by 3, then  $((k+1)^3 - (k+1))$  is divisible by 3, by Direct Proof.

[ End of Inductive Step ]

∴ For every integer  $n$  such that  $n \geq 1$ ,  $(n^3 - n)$  is divisible by 3,

by the Principle of Mathematical Induction.

Q E D

### MATHEMATICAL INDUCTION IN SYMBOLS (To View for Fun)

First Principle of Mathematical Induction (with  $a = 1$ ):

$$\left( P(1) \wedge \left[ \forall k \in \mathbb{Z}^+, P(k) \rightarrow P(k+1) \right] \right) \rightarrow \left( \forall n \in \mathbb{Z}^+, P(n) \right)$$

Second Principle of Mathematical Induction (with  $a = 1$ ):  
(Strong Mathematical Induction)

$$\left( P(1) \wedge \left[ \forall k \in \mathbb{Z}^+, \left[ \forall m \in \mathbb{Z}^+ \text{ such that } 1 \leq m \leq k, P(m) \right] \rightarrow P(k+1) \right] \right) \rightarrow \left( \forall n \in \mathbb{Z}^+, P(n) \right)$$



The PROOF OF PROPOSITION 5.3.2 in Dr. SHIRLEY'S FORMAT.

Proposition 5.3.2: For all integers  $n \geq 3$ ,  $2n+1 < 2^n$ .

Proof: [By Mathematical Induction]

Let  $n=3$ .  $2n+1 = 2 \times 3 + 1 = 7$ , by substitution.  
 $2^n = 2^3 = 8$ , by substitution.  
 $7 < 8$ . [END OF BASIS STEP]

$\therefore$  For  $n=3$ ,  $2n+1 < 2^n$ , by substitution.

Now, Suppose that  $k$  is any integer with  $k \geq 3$ .

Suppose that  $2k+1 < 2^k$ . [The Inductive Hypothesis]

[NTS:  $2(k+1)+1 < 2^{(k+1)}$ ]

$$\begin{aligned} \therefore 2(k+1)+1 &= 2k+2+1 \\ &= (2k+1)+2 \end{aligned}$$

Since  $k \geq 3$ ,  $2 < 2^k$ .

By the Inductive Hypothesis,  $2k+1 < 2^k$ .

$$\therefore (2k+1)+2 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

$$\therefore (2k+1)+2 < 2^{k+1}$$

$\therefore 2(k+1)+1 < 2^{k+1}$ , by substitution.

$\therefore$  By Direct Proof, For all integers  $k$ ,  $k \geq 3$ ,

If  $2k+1 < 2^k$ , Then  $2(k+1)+1 < 2^{k+1}$ .

[END OF Inductive Step].

$\therefore$  For all integers  $n \geq 3$ ,  $2n+1 < 2^n$ , by MATHEMATICAL INDUCTION.

QED

WORK SPACE:

$$\begin{aligned} 2(k+1)+1 &= 2k+2+1 \\ &= 2k+3 \\ &= (2k+1)+2 \\ &< 2^k + 2 \\ &< 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

EXAMPLE PROOF BY MATHEMATICAL INDUCTION -  
PROBLEM # 11 of section 5.2 SOLUTION

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TO PROVE:  $1^3 + 2^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$ , for all integers  $n \geq 1$ .

PROOF: [by Mathematical Induction]

[BASIS STEP]

$$\text{let } n=1. \quad \therefore 1^3 + 2^3 + \dots + n^3 = 1^3 = 1.$$

$$\left[ \frac{n(n+1)}{2} \right]^2 = \left[ \frac{1(1+1)}{2} \right]^2 = 1, \text{ by substitution.}$$

$$\therefore 1^3 + 2^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2 \text{ for } n=1.$$

[END OF BASIS STEP]

[INDUCTIVE STEP]

Let  $k$  be any integer such that  $k \geq 1$ .

$$\text{Suppose } 1^3 + 2^3 + \dots + k^3 = \left[ \frac{k(k+1)}{2} \right]^2. \quad \text{[THE INDUCTIVE HYPOTHESIS]}$$

$$\text{[We N.T.S. } 1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \left[ \frac{(k+1)((k+1)+1)}{2} \right]^2.]$$

$$\therefore (1^3 + 2^3 + \dots + k^3) + (k+1)^3 = \left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3$$

by the Inductive Hypothesis and Substitution,

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)(k+1)^2}{4}$$

$$\begin{aligned}
\therefore (1^3 + 2^3 + \dots + k^3) + (k+1)^3 &= \frac{(k^2 + 4(k+1))(k+1)^2}{4} && \text{(P.2)} \\
&= \frac{(k^2 + 4k + 4)(k+1)^2}{4} \\
&= \frac{(k+2)^2(k+1)^2}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
\therefore 1^3 + 2^3 + \dots + (k+1)^3 &= \left[ \frac{(k+1)(k+2)}{2} \right]^2 = \left[ \frac{(k+1)((k+1)+1)}{2} \right]^2.
\end{aligned}$$

$\therefore$  By Direct Proof, For all integers  $k \geq 1$ ,  
if  $1^3 + 2^3 + \dots + k^3 = \left[ \frac{k(k+1)}{2} \right]^2$ ,

$$\text{then } 1^3 + 2^3 + \dots + (k+1)^3 = \left[ \frac{(k+1)((k+1)+1)}{2} \right]^2$$

[GND-of- Inductive Step]

$$\therefore \text{ For all integers } n \geq 1, \quad 1^3 + 2^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2,$$

by the Principle of Mathematical Induction.

Q.E.D.